Bott characteristics of cell complexes

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ABSTRACT

A uniqueness theorem is proved for the combinatorial invariants of cell complexes defined by R. Bott in [B].

§ 0. Let $\mathscr{C}\mathscr{C}^n$ ($n \geq 0$) denote the class of all finite, at most n-dimensional, (convex) cell complexes. For any (convex) cell C, denote by $\mathring{C}$ its interior, by $\bar{C}$ the complex consisting of all faces of C, and by $\dot{C} = \bar{C} - \{C\}$ the boundary complex of C. Two complexes $K_1, K_2 \in \mathscr{C}\mathscr{C}^n$ are said to be combinatorially equivalent if there exist two complexes $L_1, L_2 \in \mathscr{C}\mathscr{C}^n$ such that: (a) K_i and L_i are combinatorially isomorphic, $i = 1, 2$; (b) L_1 and L_2 admit a common subdivision L. A combinatorial invariant on $\mathscr{C}\mathscr{C}^n$ is a function

$$\kappa : \mathscr{C}\mathscr{C}^n \to \mathbb{R},$$

such that $\kappa(K_1) = \kappa(K_2)$ whenever K_1 and K_2 are combinatorially equivalent.

Let K be an arbitrary complex in $\mathscr{C}\mathscr{C}^n$ ($n > 0$) and let C be a k-cell of K with $0 < k \leq n$. A complex K' is said to be obtained from K by an elementary bisection at C if

$$K' = (K - \{C\}) \cup \{C_1, D, C_2\},$$

where: (a) $\dim C_1 = \dim C_2 = k$; (b) $\dim D = k - 1$; (c) $\dot{D}$ is a subcomplex of $\dot{C}$; (d) $\mathring{C}_1$, $\mathring{D}$, $\mathring{C}_2$ are pairwise disjoint; (e) $\mathring{C} = \mathring{C}_1 \cup \mathring{D} \cup \mathring{C}_2$. A well known theorem (see for example [R; p. 39], [L2; p. 168], and [S; p. 34]) shows that two complexes K_1 and K_2 are combinatorially equivalent if and only if there exist two complexes L_1 and L_2 such that: (a) K_i and L_i are combinatorially isomorphic, $i=1,2$; (b) L_1 and L_2 admit a common subdivision L which can be obtained from both L_1 and L_2 by a finite sequence of elementary bisections. It follows therefore that a function $\kappa : \mathscr{C}\mathscr{C}^n \to \mathbb{R}$ is a combinatorial invariant on $\mathscr{C}\mathscr{C}^n$ if and only if $\kappa(K')=\kappa(K)$ whenever K' is obtained from K by an elementary bisection at some cell $C \in K$.

In [B] R. Bott defined two polynomials whose coefficients turn out to be combinatorial invariants on $\mathscr{C}\mathscr{C}^n$. In this paper we prove for these invariants a uniqueness result similar to that obtained by W. Mayer [M; Theorem (4.1)] and Lee Ke-chun [L1; Satz 1] for the classical Euler characteristic.

§ 1. Let $\#$ denote the cardinal number function. For a complex $K \in \mathscr{C}\mathscr{C}^n$ define:

$S(K)=$ the set of n-cells of K;
$a(K)=\#(S(K))=$ the number of n-cells of K;
$b_i(K)=$ the ith Betti number of K, $i \in \mathbb{N}$;
$\mathscr{P}_k^r(K)=\{P \subset S(K) \mid \#(P)=k+1,\ b_n(K-P)=r\}$, $k,r \in \mathbb{Z}$;
$p_k^r(K)=\#(\mathscr{P}_k^r(K))$, $k,r \in \mathbb{Z}$;
$B^r(K)=\sum_{k\geq 0}(-1)^k p_k^r(K)$, $r \in \mathbb{Z}$;
$\tilde{B}^r(K)=\sum_k(-1)^k p_k^r(K)$, $r \in \mathbb{Z}$.

The numbers $B^r(K)$ and $\tilde{B}^r(K)$ will be called, respectively, the rth Bott characteristic of K and the rth reduced Bott characteristic of K.

It is easy to see that $p_k^r(K)\neq 0$ implies

(a) $\qquad -1 \leq k \leq a(K)-1$

and

(b) $\qquad \max(0, b_n(K)-k-1) \leq r \leq b_n(K)$.

From (a) it follows that $B^r(K)$ and $\tilde{B}^r(K)$ are well defined for any $r \in \mathbb{Z}$; on the other hand, (b) shows that both $B^r(K)\neq 0$ and $\tilde{B}^r(K)\neq 0$ imply $0 \leq r \leq b_n(K)$. Two well known combinatorial identities show that

(c) $\qquad \sum_{r,k} p_k^r(K)=\sum_k\left(\sum_r p_k^r(K)\right)=2^{a(K)}$

and

(d) $\qquad \sum_r \tilde{B}^r(K)=\sum_k(-1)^k\left(\sum_r p_k^r(K)\right)=0.$

The problem which arises is to determine which linear combinations of the numbers $p_k^r(K)$, $k,r \in \mathbb{Z}$, are combinatorial invariants on $\mathscr{C}\mathscr{C}^n$; identity (c) shows that there are such combinations which are not combinatorial invariants. Here we get the main result of the paper ($\delta_{r,s}$ denotes, as usually, the Kronecker symbol).

1.1. THEOREM. If a function

$$\kappa = \sum_{r,k} \alpha_k^r p_k^r : \mathscr{C}\mathscr{C}^n \to \mathbb{R},$$

with $\alpha_k^r \in \mathbb{R}$ for $k, r \in \mathbb{Z}$, is a combinatorial invariant on $\mathscr{C}\mathscr{C}^n$ $(n>0)$, then

$$\begin{aligned}\kappa(K) &= \sum_r \alpha_{-1}^r \delta_{r, b_n(K)} + \sum_r \alpha_0^r B^r(K) = \\ &= \sum_r (\alpha_{-1}^r + \alpha_0^r)\delta_{r, b_n(K)} + \sum_r \alpha_0^r \tilde{B}^r(K)\end{aligned}$$

for each $K \in \mathscr{C}\mathscr{C}^n$.

PROOF. Let K be an n-complex. Each cell $C \in S(K)$ defines a partition $\{\mathscr{Q}_{k;0}^r(K;C), \mathscr{Q}_{k;1}^r(K;C)\}$ of $\mathscr{P}_k^r(K)$, where

$$\mathscr{Q}_{k;0}^r(K;C) = \{P \in \mathscr{P}_k^r(K) | C \notin P\}$$

and

$$\mathscr{Q}_{k;1}^r(K;C) = \{P \in \mathscr{P}_k^r(K) | C \in P\}.$$

There is an obvious bijection

$$\mathscr{Q}_{k;1}^r(K;C) \to \mathscr{P}_{k-1}^r(K - \{C\}), \qquad P \mapsto P - \{C\};$$

therefore,

$$p_k^r(K) = \#(\mathscr{Q}_{k;0}^r(K;C)) + p_{k-1}^r(K - \{C\}).$$

On the other hand, if $K' = (K - \{C\}) \cup \{C_1, D, C_2\}$ is the subdivision of K obtained by an elementary bisection at C, then a partition $\{\mathscr{Q}_{k;0,0}^r(K;C), \mathscr{Q}_{k;1,0}^r(K;C), \mathscr{Q}_{k;0,1}^r(K;C), \mathscr{Q}_{k;1,1}^r(K;C)\}$ of $\mathscr{P}_k^r(K')$ is defined, where

$$\mathscr{Q}_{k;0,0}^r(K;C) = \{P \in \mathscr{P}_k^r(K') | C_1 \notin P,\ C_2 \notin P\},$$

$$\mathscr{Q}_{k;1,0}^r(K;C) = \{P \in \mathscr{P}_k^r(K') | C_1 \in P,\ C_2 \notin P\},$$

$$\mathscr{Q}_{k;0,1}^r(K;C) = \{P \in \mathscr{P}_k^r(K') | C_1 \notin P,\ C_2 \in P\},$$

and

$$\mathscr{Q}_{k;1,1}^r(K;C) = \{P \in \mathscr{P}_k^r(K') | C_1 \in P,\ C_2 \in P\}.$$

Obviously, $\mathscr{Q}_{k;0,0}^r(K;C) = \mathscr{Q}_{k;0}^r(K;C)$. Further, there are bijections

$$\mathscr{Q}_{k;1,0}^r(K;C) \to \mathscr{P}_{k-1}^r(K - \{C\}), \qquad P \mapsto P - \{C_1\},$$

$$\mathscr{Q}_{k;0,1}^r(K;C) \to \mathscr{P}_{k-1}^r(K - \{C\}), \qquad P \mapsto P - \{C_2\},$$

and

$$\mathscr{Q}_{k;1,1}^r(K;C) \to \mathscr{P}_{k-2}^r(K - \{C\}), \qquad P \mapsto P - \{C_1, C_2\}.$$

To prove the first one, it is sufficient to note that, for $P \in \mathscr{Q}_{k;1,0}^r(K;C)$, the complex $K' - P = L \cup \{D, C_2\}$, where $L = K - \{C\} - (P - \{C_1\})$, collapses to L; the second one is obtained in a similar manner. For the third bijection, observe that, for $P \in \mathscr{Q}_{k;1,1}^r(K;C)$, we have $K' - P = L \cup \bar{D}$ and $L \cap \bar{D} = \dot{D}$, where

$L=K-\{C\}-(P-\{C_1,C_2\})$; a simple Mayer-Vietoris argument shows that $b_n(K'-P)=b_n(L)$. We get, therefore,

$$p_k^r(K')=\#(\mathscr{Q}_{k;0}^r(K;C))+2p_{k-1}^r(K-\{C\})+p_{k-2}^r(K-\{C\}).$$

The two formulae derived above imply that

$$(F)\qquad p_k^r(K')-p_k^r(K)=p_{k-1}^r(K-\{C\})+p_{k-2}^r(K-\{C\}).$$

Let now

$$\kappa=\sum_r\sum_k\alpha_k^r p_k^r,$$

with $\alpha_k^r\in\mathbb{R}$ for $k,r\in\mathbb{Z}$, be a combinatorial invariant on $\mathscr{C}\mathscr{C}^n$. Then $\kappa(K')=$ $=\kappa(K)$ whenever K' is obtained from K by an elementary bisection at some $C\in S(K)$. From formula (F) we get

$$\kappa(K')-\kappa(K)=\sum_r\sum_k\alpha_k^r(p_{k-1}^r(K-\{C\})+p_{k-2}^r(K-\{C\}))$$

and this implies

$$(G)\qquad \sum_r\sum_k(\alpha_{k+1}^r+\alpha_{k+2}^r)p_k^r(K-\{C\})=0.$$

For each $k,r\in\mathbb{Z}$, $k\geq-1$, $r\geq0$, let

$$W_k^r=\bar{s}\cup\bar{s}_0\cup\bar{s}_1\cup\ldots\cup\bar{s}_k\cup\dot{S}_1\cup\ldots\cup\dot{S}_r$$

be a disjoint union of complexes, where s is an n-simplex, each s_i, $i=0,\ldots,k$, is an n-simplex, and each S_i, $i=1,\ldots,r$, is an $(n+1)$-simplex. Obviously, $b_n(W_k^r)=r$; this implies that $p_i^m(W_k^r-\{s\})=0$ for all $i\in\mathbb{Z}$ and $m>r$. If $P\in\mathscr{P}_k^r(W_k^r-\{s\})$, then necessarily $P=\{s_0,\ldots,s_k\}$ (because deleting an n-simplex from one of the r n-spheres would lower the value of b_n); it follows that $p_k^r(W_k^r-\{s\})=1$. For similar reasons, $p_i^r(W_k^r-\{s\})=0$ for all $i>k$. From formula (G) we get for all $k\geq-1$ and $r\geq0$

$$(I_k^r)\qquad \sum_m\sum_i(\alpha_{i+1}^m+\alpha_{i+2}^m)p_i^m(W_k^r-\{s\})=0.$$

Now a double inductive argument will be applied. For $r=0$ we have

$$(I_k^0)\qquad \sum_i(\alpha_{i+1}^0+\alpha_{i+2}^0)p_i^0(W_k^0-\{s\})=0;$$

in particular,

$$(I_{-1}^0)\qquad \alpha_0^0+\alpha_1^0=0.$$

If we assume that $\alpha_i^0+\alpha_{i+1}^0=0$ for $i=0,\ldots,k$, then (I_k^0) implies $\alpha_{k+1}^0+\alpha_{k+2}^0=0$. Therefore, $\alpha_k^0+\alpha_{k+1}^0=0$ for all $k\geq0$. Further, assume that for $m=0,\ldots,r-1$,

$$\alpha_k^m+\alpha_{k+1}^m=0,\text{ for all }k\geq0.$$

Formula (I_k^r) then implies that

$$\sum_i(\alpha_{i+1}^r+\alpha_{i+2}^r)p_i^r(W_k^r-\{s\})=0,$$

for all $k \geq -1$; induction on k shows that $\alpha_k^r + \alpha_{k+1}^r = 0$ for all $k \geq 0$. Thus we have proved that

$$\alpha_k^r + \alpha_{k+1}^r = 0,$$

for all $k \geq 0$ and $r \geq 0$. It follows that for all $r \geq 0$,

$$\alpha_k^r = (-1)^k \alpha_0^r, \; k \geq 0.$$

It still remains to note that

$$p_{-1}^r(K) = \delta_{r, b_n(K)}, \; r \geq 0,$$

and we obtain

$$\begin{aligned}\kappa(K) &= \textstyle\sum_r (\alpha_{-1}^r \delta_{r, b_n(K)} + \alpha_0^r \sum_{k \geq 0} (-1)^k p_k^r(K)) = \\ &= \textstyle\sum_r ((\alpha_{-1}^r + \alpha_0^r)\delta_{r, b_n(K)} + \alpha_0^r \sum_k (-1)^k p_k^r(K)),\end{aligned}$$

for all $K \in \mathscr{C}\mathscr{C}^n$. □

1.2. COROLLARY. If a function

$$\kappa = \textstyle\sum_r \sum_{k \geq 0} \alpha_k^r p_k^r : \mathscr{C}\mathscr{C}^n \to \mathbb{R},$$

with $\alpha_k^r \in \mathbb{R}$ for $k, r \in \mathbb{Z}$, is a combinatorial invariant on $\mathscr{C}\mathscr{C}^n$ $(n > 0)$, then

$$\kappa(K) = \textstyle\sum_r \alpha_0^r B^r(K) = \sum_r \alpha_0^r \delta_{r, b_n(K)} + \sum_r \alpha_0^r \tilde{B}^r(K)$$

for each $K \in \mathscr{C}\mathscr{C}^n$.

1.3. COROLLARY. If a function

$$\kappa = \textstyle\sum_k \alpha_k p_k^r : \mathscr{C}\mathscr{C}^n \to \mathbb{R},$$

with $\alpha_k \in \mathbb{R}$ for $k \in \mathbb{Z}$, and fixed $r \geq 0$, is a combinatorial invariant on $\mathscr{C}\mathscr{C}^n$ $(n > 0)$, then

$$\begin{aligned}\kappa(K) &= \alpha_{-1}\delta_{r, b_n(K)} + \alpha_0 B^r(K) = \\ &= (\alpha_{-1} + \alpha_0)\delta_{r, b_n(K)} + \alpha_0 \tilde{B}^r(K)\end{aligned}$$

for each $K \in \mathscr{C}\mathscr{C}^n$.

1.4. COROLLARY. If a function

$$\kappa = \textstyle\sum_{k \geq 0} \alpha_k p_k^r : \mathscr{C}\mathscr{C}^n \to \mathbb{R},$$

with $\alpha_k \in \mathbb{R}$ for $k \in \mathbb{Z}$, and fixed $r \geq 0$, is a combinatorial invariant on $\mathscr{C}\mathscr{C}^n$ $(n > 0)$, then

$$\kappa(K) = \alpha_0 B^r(K) = \alpha_0 \delta_{r, b_n(K)} + \alpha_0 \tilde{B}^r(K)$$

for each $K \in \mathscr{C}\mathscr{C}^n$.

§ 2. REMARKS

2.1. For $n = 0$, the numbers $p_k^r(K)$ can be defined by using the reduced

Betti numbers $\tilde{b}_0(K-P)$ instead of the unreduced ones. With each of these two definitions, the result stated in Theorem 1.1. turns out to be false; in fact, every function

$$\kappa = \sum_{r,k} \alpha_k^r p_k^r : \mathscr{C}\mathscr{C}^0 \to \mathbb{R}$$

is a combinatorial invariant on $\mathscr{C}\mathscr{C}^0$.

2.2. The first polynomial defined in [B] is

$$\mathscr{R}(K;\lambda) = \sum_{P\subset S(K)} (-1)^{\#(P)} \lambda^{b_n(K-P)}.$$

This can be written as

$$\mathscr{R}(K;\lambda) = \sum_r \sum_k \sum_{P\in \mathscr{P}_k^r(K)} (-1)^{k+1} \lambda^r =$$
$$= -\sum_r \left(\sum_k (-1)^k p_k^r(K)\right)\lambda^r = -\sum_r \tilde{B}^r(K)\lambda^r.$$

The main result in that paper is in fact the combinatorial invariance of the reduced Bott characteristics $\tilde{B}^r(K)$, $r\geq 0$.

The second polynomial defined in [B] is

$$\mathscr{S}(K;\lambda) = \sum_{P\subset S(K)} i^{\#(P)+b_{n-1}(K-P)} \lambda^{b_n(K-P)} \quad (i=\sqrt{-1}).$$

It is easy to see that

$$(-1)^{n-1} b_{n-1}(K) + (-1)^n b_n(K) =$$
$$= (-1)^{n-1} b_{n-1}(K-P) + (-1)^n b_n(K-P) + (-1)^n \#(P);$$

from this we get

$$b_{n-1}(K-P) = b_{n-1}(K) - b_n(K) + b_n(K-P) + \#(P).$$

Now we can write

$$\mathscr{S}(K;\lambda) = \sum_{P\subset S(K)} i^{2\#(P)+b_{n-1}(K)-b_n(K)+b_n(K-P)} \lambda^{b_n(K-P)} =$$
$$= i^{b_{n-1}(K)-b_n(K)} \sum_{P\subset S(K)} (-1)^{\#(P)} (\lambda i)^{b_n(K-P)};$$

this shows that

$$\mathscr{S}(K;\lambda) = i^{b_{n-1}(K)-b_n(K)} \cdot \mathscr{R}(K;\lambda i).$$

The relation above proves that the $\mathscr{S}$ polynomial does not provide any more information than the $\mathscr{R}$ polynomial and the Betti numbers.

2.3. For each complex $K\in \mathscr{C}\mathscr{C}^n$ $(n\geq 0)$ denote

$$T(K) = \{C\in S(K) \mid b_n(K-\{C\}) = b_n(K)\}.$$

A simple Mayer-Vietoris argument shows that if $P=P_0\cup P_1$ and $P_0\cap P_1=\varnothing$, where $P_0\subset T(K)$ and $P_1\subset S(K)-T(K)$, then $b_n(K-P)=b_n(K-P_1)$; in particular, $b_n(K-P)=b_n(K)$ for each $P\subset T(K)$.

If $T(K)\neq\varnothing$, let $C\in T(K)$ be a fixed cell. Then for each $k,r\in\mathbb{Z}$,

$$p_k^r(K) = p_k^r(K-\{C\}) + p_{k-1}^r(K-\{C\});$$

therefore, for each $r \in \mathbb{Z}$,

$$\tilde{B}^r(K) = \sum_k (-1)^k (p_k^r(K-\{C\}) + p_{k-1}^r(K-\{C\})) = 0.$$

From this we get $B^{b_n(K)}(K) = 1$. On the other hand, if $T(K) = \varnothing$, we get $\tilde{B}^{b_n(K)}(K) = -1$ and $B^{b_n(K)}(K) = 0$.

2.4. An interpretation of the reduced Bott characteristics $\tilde{B}^r(K)$, $r \geq 0$, as the Euler characteristics of some simplicial pairs is obtained through the following construction. Each complex $K \in \mathscr{C}\mathscr{C}^n$ defines a simplex $\Delta(K)$ whose vertices are the cells $C \in S(K)$; the k-simplices of $\overline{\Delta(K)}$ are then the sets $P \in \bigcup_r \mathscr{P}_k^r(K)$. A sequence $F_r(K)$, $r \geq 0$, of subcomplexes of $\overline{\Delta(K)}$ is determined, where

$$F_r(K) = \bigcup_{m \geq r} \bigcup_k \mathscr{P}_k^r(K),\ r \geq 0.$$

Then

$$F_r(K) - F_{r+1}(K) = \bigcup_k \mathscr{P}_k^r(K),\ r \geq 0,$$

and

$$\tilde{B}^r(K) = \chi(F_r(K), F_{r+1}(K)),\ r \geq 0.$$

This implies

$$B^r(K) = \chi(F_r(K), F_{r+1}(K))$$

for $r < b_n(K)$, and $B^{b_n(K)}(K) = \chi(F_{b_n(K)}(K))$.

REFERENCES

[B] Bott, Raoul – Two new combinatorial invariants for polyhedra. Portugaliae Math. **11**, 35–40 (1952). MR 14, 74; Zbl. 47, 420.

[L1] Lee Ke-chun – Über die Eindeutigkeit von einigen kombinatorischen Invarianten endliches Komplexes. Sci. Record (N.S.) **1**, 279–281 (1957). MR 20#6089; Zbl. 90, 389.

[L2] Lefschetz, Solomon – Algebraic Topology. American Mathematical Society Colloquium Publications, **27**. American Mathematical Society, New York, 1942. MR 4, 84; Zbl. 61, 393. Reprinted: 1948. Zbl. 36, 122.

[M] Mayer, W. – A new homology theory. II. Ann. of Math. **43**, 594–605 (1942). MR 3, 318; Zbl. 61, 403.

[R] Reidemeister, Kurt – Topologie der Polyeder und kombinatorische Topologie der Komplexe. Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, Band 17. Akademische Verlagsgesellschaft, Leipzig, 1938. Zbl. 19, 186. Reprinted: J.W. Edwards, Ann Arbor, Michigan, 1944. MR 6, 97. 2te Aufl. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Bd 17. Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1953. MR 15, 457.

[S] Stallings, John R. – Lectures on polyhedral topology. Notes by Ananda Swarup. Tata Institute of Fundamental Research Lectures on Mathematics, No. 43. Tata Institute of Fundamental Research, Bombay, 1967. MR 38#6605; Zbl. 182, 262.